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The Global Theory of Paths in Networks. I. Definitions, Examples and Limits

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0.0. INTRODUCTION

This paper initiates the study of minimum path problems from the global point of view. It may be regarded as a variation on the theme introduced in [10]. That paper defined several notions of morphism inherent in the max-flow problem of Ford–Fulkerson and studied the resulting categories. In this paper we define the corresponding notions of morphism inherent in the minimum path problem and study the categories which result from them. As one would expect, this same approach to problems already known to have common features do produce similar results, but the details are often strikingly different and the similarities sometimes unexpected.

0.1. *Basic Definitions*

A *directed graph*, G, consists of a set V of vertices, a set E of edges, and a pair of functions, $\partial_+, \partial_- : E \to V$, which identify the head and tail end, respectively, of each edge. Extend $\partial_\pm$ to set functions by $\partial_\pm(A) = \{\partial_\pm(e) : e \in A\}$ and let $\partial = \partial_+ \cup \partial_-$. Note that every directed graph is the image of a functor whose domain is the diagram category

$$\begin{tikzcd} E \arrow[r, bend left, "\partial_+"] \arrow[r, bend right, "\partial_-"'] & V \end{tikzcd}$$

and whose codomain is the category SET of (finite) sets. The obvious notion of morphism for directed graphs G and H is a pair of functions $\varphi_E : E_G \to E_H$ and $\varphi_V : V_G \to V_H$ such that the diagrams

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$$\begin{tikzcd} E_G \arrow[r, "\partial \pm"] \arrow[d, "\varphi_E"'] & V_G \arrow[d, "\varphi_V"] \\ E_H \arrow[r, "\partial_\pm"'] & V_H \end{tikzcd}$$

commute. φ_E and φ_V are then the components of a natural transformation from the functor defining G to that defining H. Thus DIGRAPH, the category of directed graphs with these morphisms, is the functor category FUNCT($\cdot \rightrightarrows \cdot$, SET).

A *network*, N, consists of a directed graph, G, a weight function $\omega : V \to \mathbb{R}^+$, and two distinguished vertices s and t (not necessarily distinct), s will be called the *initial* vertex and t the *terminal* one. An *s–t path* in N is a sequence of edges $e_1, \ldots, e_k$, such that

(i) $\partial_-(e_1) = s$ and $\partial_+(e_k) = t$ and

(ii) for all i, $1 \leq i \leq k-1$, $\partial_+(e_i) = \partial_-(e_{i+1})$.

Let $\mathcal{P}(N)$ be the set of all s–t paths in N. The *weight* of $P \in \mathcal{P}(N)$ is the sum of the weights of its vertices, $\omega(P) = \Sigma_{v \in \partial(P)} \omega(v)$. The *minimum path problem* (MPP) is: Given a network, N, compute $\lambda(N) = \mathrm{Min}_{P \in \mathcal{P}(N)} \omega(P)$, the minimum weight of any s–t path in N, and find a path whose weight is this minimum.

EXAMPLE. (i) If N has no s–t paths, then $\mathcal{P}(N) = \phi$ and $\lambda(N) = \infty$.

(ii) if N is a linear network,

$$x_0 = s \cdot \xrightarrow{e_1} \underset{x_1}{\cdot} \xrightarrow{e_2} \cdot \;\cdots\cdot\; \underset{x_{k-1}}{\cdot} \xrightarrow{e_k} \cdot\, x_k = t,$$

then $\lambda(N) = \Sigma_{i=0}^{k} \omega(x_i)$.

(iii) If N is the network in Fig. 1 then $\lambda(N) = 4$.

Note. For technical convenience we have put weights on vertices, but we could also formulate the MPP with edge weights (in fact that is the standard way) or both edge and vertex weights. There is no essential difference in the resulting theories and we may apply our results to digraphs with any of these weightings.

0.2. *Some Background on the Minimum Path Problem*

The MPP has long been known to be solvable in polynomial time [4, 11]. The fastest algorithm for it is the one which Dijkstra derived from the

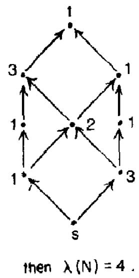


FIGURE 1

celebrated "sproing" heuristic [3]. If $|V| = n$ and $\Sigma_{v \in V} \omega(v) = W$, then Dijkstra's algorithm runs in time $O(n^2 \ln W)$.

Given any undirected graph $G = (V, E)$, $|V| = n$, a *numbering* of G is a function $\varphi : V \to \{1, \ldots, n\}$ which is one-to-one and onto. For any numbering φ of G and edge $e \in E$, let $\Delta_e(\varphi) = |\varphi(v) - \varphi(w))|$, where v and w are the vertices incident to e. The *edge-sum problem* (ESP) then is to minimize $\Sigma_{e \in E} \Delta_e(\varphi)$ over all numberings, φ, of G. However, if, given a numbering, φ, we let $S_l(\varphi) = \{v \in V: \varphi(v) \leq l\}$ for $0 \leq l \leq n$ and for all $S \subseteq V$ we define $\omega(S) = |\{e \in E : e \text{ is incident to } v \in S \text{ and } w \notin S\}|$ then we have

LEMMA. (see [6]). $\Sigma_{e \in E} \Delta_e(\varphi) = \Sigma_{l=0}^{m} \omega(S_l(\varphi))$.

The lemma shows that given an instance of $G = (V, E)$ of the ESP, if we form a network, $N(G)$ with vertex set $V' = 2^V$, the power set of V, edgeset $E' = \{(A, B) : A \subseteq B \subseteq V, |B| = |A| + 1\}$, with $\partial_-(A, B) = A$, $\partial_+(A, B) = B$, and distinguished vertices $s = \phi$, $t = V$ then the ESP on G is equivalent to the MPP on $N(G)$. This is not a polynomial reduction of course, since $|2^V| = 2^{|V|}$, but it is simple in other ways and has been useful in solving a number of special cases of the ESP [6, 7, 9].

EXAMPLE 1. Let G be the graph of the square in Fig. 2a. $|V| = 4$ so $|V'| = 2^4 = 16$. $N(G)$ is then as shown in Fig. 2b.

The ESP is known to be NP-complete [5] and the hitherto best algorithm known was the brute force algorithm which compared $\Sigma_{e \in E} \Delta_e(\varphi)$ for all $n!$ possible numberings. However, constructing $N(G)$ and solving the MPP on

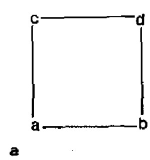


FIGURE 2a

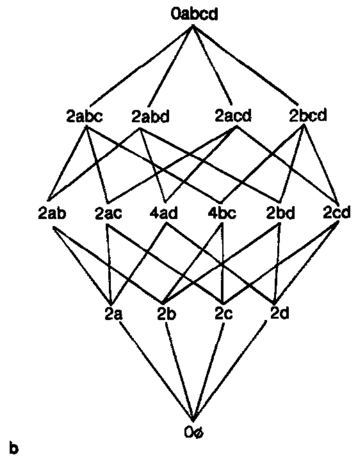


FIG. 2b.—All edges directed upward. The number to the left of each vertex is its weight.

$N(G)$ by Dijkstra's algorithm requires only $O((2^n)^2 \ln n^2) = O(2^{2n} \ln n)$. Since $\ln n! \simeq n \ln n$ by Sterling's formula, and $\ln(2^{2n} \ln n) \simeq 2n$ we see that a considerable savings has been effected. This is of some theoretical interest, but $O(2^{2n} \ln n)$ is still exponential so the reduction would only be feasible for small n unless some way could be found to reduce the size of N. This leads us to the concept of path-morphism.

1. Path-Morphisms

Given networks M and N, a *partial path-morphism* $\varphi: M \to N$ is a directed graph homomorphism such that

(i) $\varphi_V^{-1}(s_N) = s_M$ and $\varphi_V^{-1}(t_N) = t_M$ and

(ii) for all $v \in V_M$, $\omega_M(v) \geq \omega_N(\varphi_V(v))$.

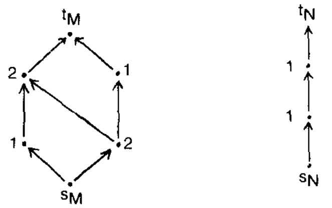
Recall from Section 0.1 that $\mathcal{P}(N)$ is the set of all s–t paths in N. It is easily verified that a partial path-morphism $\varphi: M \to N$ induces a function $\mathcal{P}(\varphi): \mathcal{P}(M) \to \mathcal{P}(N)$ which is weight decreasing, i.e., for all $P \in \mathcal{P}(M)$, $\omega_M(P) \geq \omega_N((\varphi)[P])$. Therefore

$$\lambda(M) = \operatorname*{Min}_{P \in \mathcal{P}(M)} \omega_M(P) \geq \operatorname*{Min}_{P' \in \mathcal{P}(N)} \omega_N(P') = \lambda(N);$$

The weights on V_N should naturally be made as large as possible, consistent with having $\omega_M(v) \geq \omega_N(\varphi_V(v))$ for all $v \in V_M$. This means that for $v' \in V_N$ we should have $\omega_N(v') = \min\{\omega_M(v) : \varphi_V(v) = v'\}$. A partial path-morphism which has this property for all $v' \in V_N$ is called *sharp*.

Sharpness, however, is still not enough to ensure that $\lambda(M) = \lambda(N)$, as the following example shows:

EXAMPLE 2.



$\lambda(M) = 3$ $\lambda(N) = 2$

It would be nice to have natural conditions in the partial path-morphism φ which would make the inequality $\lambda(M) \geq \lambda(N)$ an equality. After some consideration of the possibilities for such a definition in the light of examples and the resulting theories, we arrived at the following definition as the most natural and useful: A partial path-morphism $\varphi : M \to N$ is called a *path-morphism* if (i) φ is sharp, and

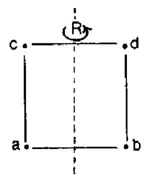
(ii) for all $e' \in E_N$ and $v \in V_M$ such that $\varphi_V(v) = \partial_-(e')$ and $\omega_M(v) = \omega_N(\partial_-(e'))$ there is an $e \in E_M$ such that $\partial_-(e) = v$, $\varphi_E(e) = e'$ and $\omega_M(\partial_+(e)) = \omega_N(\partial_+(e'))$. This definition may be paraphrased as saying that φ^{-1} preserves minimum paths "locally." Note that a path-morphism must be onto, i.e., it is a digraph epimorphism.

FUNDAMENTAL LEMMA. *If $\varphi : M \to N$ is a path-morphism then $\mathcal{P}(\varphi) : \mathcal{P}(M) \to \mathcal{P}(N)$ has a weight-preserving right inverse $\rho : \mathcal{P}(N) \to \mathcal{P}(M)$. Therefore $\lambda(M) = \lambda(N)$.*

Proof. Given $P' = \{e'_1, \ldots, e'_k\} \in \mathcal{P}(N)$ we construct $\rho(P') = \{e_1, \ldots, e_k\} \in \mathcal{P}(M)$ one edge at a time: since s_M is the only vertex which φ maps onto s_N and since φ is sharp, we must have $\omega_M(s_M) = \omega_N(s_N)$. Of course $\partial_-(e'_1) = s_N$ so by part (ii) of the definition of path-morphism there exists $e_1 \in E_M$ such that $\partial_-(e_1) = s_M$, $\varphi_E(e_1) = e'_1$ and $\omega_M(\partial_+(e_1)) = \omega_N(\partial_+(e'_1))$. We then repeat the process with $\partial_+(e_1)$ in place of s_M to obtain e_2 and so on until we reach e_k. Since t_M is the only vertex mapped to t_N by φ, $\partial_+(e_k) = t_M$ and we are done.

The Fundamental Lemma may be paraphrased as saying that a digraph homomorphism which preserves the MPP locally preserves it globally. The definition of path-morphism was made minimal with respect to implying the Fundamental Lemma. This results in a curious asymmetry in the definition; reversing the directions of edges in condition (ii) of the definition gives a different statement. These might be called the "up" and "down" versions of the notion of path-morphism and a path-morphism which is both "up" and "down" might be called "anisotropic." All the applications so far involve "anisotropic" path-morphisms, but for simplicity and generality we shall stick to the "up" version given in the definition.

EXAMPLE 3. (i) Any symmetry of a network is a path-morphism. In particular, in Example 1, if R is a (reflective) symmetry of the square



then it induces a symmetry of the corresponding network.

(ii) In [9] it was shown how reflective symmetries (such as the one above) may be used to define an operation on $N(G)$, called a stabilizing operator. This operator is a path morphism whose image for the reflection of Example 3(i) acting on the network of Fig. 2b is shown in Fig. 3.
The values of φ are determined by

$$
\begin{aligned}
\varphi(\varnothing) &= \varnothing, \\
\varphi(\{a\}) &= \{a\} = \varphi(\{b\}), \\
\varphi(\{c\}) &= \{c\} = \varphi(\{d\}), \\
\varphi(\{a, b\}) &= \{a, b\}, \\
\varphi(\{a, c\}) &= \{a, c\} = \varphi(\{a, d\}) = \varphi(\{b, c\}), \\
\varphi(\{c, d\}) &= \{c, d\}, \\
\varphi(\{a, b, c\}) &= \{a, b, c\} = \varphi(\{a, b, d\}), \\
\varphi(\{a, c, d\}) &= \{a, c, d\} = \varphi(\{b, c, d\}).
\end{aligned}
$$

(iii) If given a network N, we can find a path-morphism $\varphi : N \to L$, L a linear network (chain), then we say that N has the K–H property.

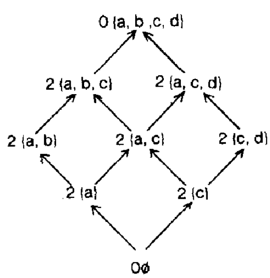


FIGURE 3

If N has the K–H property then the MPP on it is essentially solved since $\lambda(N) = \lambda(L) = \omega_L(L)$. In the above example we may take L to be

$$\begin{tikzcd}[row sep=small] 0\ \{a, b, c, d\} \\ 2\ \{a, b, c\} \arrow[u] \\ 2\ \{a, b\} \arrow[u] \\ 2\ \{a\} \arrow[u] \\ 0\ \emptyset \arrow[u] \end{tikzcd}$$

with

$$\begin{aligned} \varphi(\varnothing) &= \varnothing, \\ \varphi(\{a\}) &= \{a\} = \varphi(\{c\}), \\ \varphi(\{a, b\}) &= \{a, b\} = \varphi(\{a, c\}) = \varphi(\{c, d\}), \\ \varphi(\{a, b, c\}) &= \{a, b, c\} = \varphi(\{a, c, d\}), \\ \varphi(\{a, b, c, d\}) &= \{a, b, c, d\}. \end{aligned}$$

The K–H property might seem an unlikely occurrence, however, the networks of the graphs of the regular n-gons the simplices, cubes and cross-polytopes in all dimensions, the dodecahedron and icosahedron in three dimensions and the 24-cell in four dimensions have all been shown [9] to have the K–H property. The question of whether the networks of all regular solids have it has been unanswered only for two regular solids, the 120-cell and 600-cell in four dimensions. (See [9] and Section 2.2.)

(iv) Clearly, if $\varphi : M \to N$ is a partial path-morphism and $N \subseteq M$ (or equivalently φ has a right inverse $\rho_0 : N \to M$), then φ is a path-morphism (with $\rho = \mathcal{P}(\rho_0)$. It might seem from these examples that ρ always arises in this way, but Fig. 4 shows this is not the case: Clearly φ is a path-morphism, but ρ cannot be induced by any ρ_0 since there is no (undirected) circuit 1, 3, 2, 4, 1, in the domain.

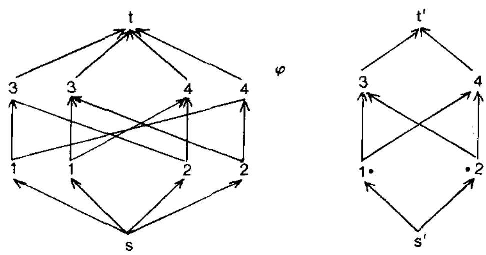


FIG. 4.—Each pair of same-numbered vertices is mapped to the same-numbered vertex at right.

1.1. *The Categories of the Minimum Path Problem*

It is easily verified that by taking networks as objects and the set of all partial path-morphisms $\varphi: M \to N$ as $\hom(M, N)$ we form a category [12]: The composition of partial path-morphisms $\varphi \in \hom(M, N)$ and $\tau \in \hom(N, P)$ is their composition, $\tau \circ \varphi$ as digraph homomorphisms. The fact that $\tau \circ \varphi: M \to P$ is a partial path-morphism follows from the transitivity of "$\geq$" in $\mathbb{R}$. Associativity and identities are thus inherited from DIGRAPH. This category we denote by *PPATH.*

Since DIGRAPH is the functor category FUNCT ($\cdot\rightrightarrows\cdot$, SET), it inherits all limits and colimits from SET (see [12, Theorem V.3.1]). These constructions may be extended to PPATH by defining

$$\omega(v') = \max\{\omega(v) : c \in Ob(C) \quad \text{and} \quad \rho_c(v') = v\} \forall v' \in V_{\varprojlim(\bar{F})}$$

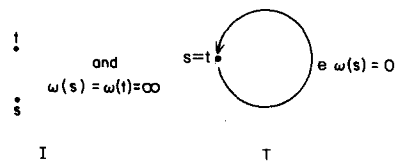
and

$$\omega(v'') = \min\{\omega(v) : c \in Ob(C) \quad \text{and} \quad \rho_c(v) = v''\} \forall v'' \in V_{\varinjlim(\bar{F})},$$

F being any (finite) functor $F: C \to$ PPATH and $\bar{F}$ being the corresponding functor $\bar{F}: C \to$ DIGRAPH. Thus we have

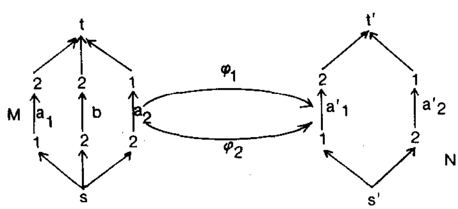
THEOREM. *PPATH has all finite limits and colimits.*

EXAMPLE 4. (i) The initial and terminal objects in PPATH are

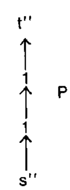


respectively. $\lambda(I) = \infty$ and $\lambda(T) = 0$

(ii) The coequalizer in PPATH of the diagram



where $\varphi_1(b) = a_1^1$, $\varphi_2(b) = a_2^1$, is



The set of all path-morphisms $\varphi : M \to N$ is also the hom-set for a category: Composition is inherited from FPATH which was inherited from DIGRAPH which was in turn inherited for SET. The fact that the composition $\rho \circ \varphi$ in PPATH of path-morphisms $\varphi : M \to N$ and $\rho : N \to P$ is a path-morphism is a simple exercise in the definition of path-morphism. Associativity of composition and identities are thus inherited from PPATH. This category we denote by *PATH*.

As we have seen, PPATH has all finite limits and colimits, inheriting them from DIGRAPH. Example 4(ii), however, shows that the situation in PATH is not quite so simple: φ_1 and φ_2 are actually path-morphisms, but their coequalizer (in PPATH), is not (as shown in Example 2). Thus limits cannot always be carried over from PPATH to PATH; at least some modification is required.

1.2. *What Can Be Said about Limits in PATH?*

A path-morphism $\varphi : M \to N$ is called *exact* if for all $v \in V_M$, $\omega_N(\varphi(v)) = \omega_M(v)$.

THEOREM. *If φ_i, $i = 1, 2$, are exact in the diagram*

$$\begin{tikzcd} M_1 \arrow[dr, "\varphi_1"] & \\ & N \\ M_2 \arrow[ur, "\varphi_2"'] & \end{tikzcd}$$

, then its limit (pullback)

$$\begin{tikzcd} & M_1 \arrow[dr, "\varphi_1"] & \\ L \arrow[ur, "\pi_1"] \arrow[dr, "\pi_2"'] & & N \\ & M_2 \arrow[ur, "\varphi_2"'] & \end{tikzcd}$$

in PPATH is inherited by PATH and the π_i, $i = 1, 2$, are exact.

Proof. Suppose $e_1 \in E_{M_1}$ and $(v_1, v_2) \in V_L$ with $\partial_-(e_1) = v_1$. Then $\varphi_1(e_1) \in E_N$, $v_2 \in V_{M_2}$, and $\partial_-(\varphi_1(e_1)) = \varphi_2(v_2)$. Since φ_2 is a path-morphism there exists $e_2 \in E_{M_2}$ such that $\partial_-(e_2) = v_2$ and $\varphi_2(e_2) = \varphi_1(e_1)$. Therefore $(e_1, e_2) \in E_L$, $\partial_-(e_1, e_2) = (v_1, v_2)$, and $\pi_1(e_1, e_2) = e_1$.

Examples show the necessity of exactness in this theorem.

PATH cannot have a terminal object T since that would imply $\lambda(N) = \lambda(T)$, a constant, for all networks. Equalizers are not inherited from PPATH, even for exact path-morphisms, since they are not generally onto. Now, turning to colimits we have

THEOREM. *If φ_i, $i = 1, 2$, are exact in the diagram*

$$\begin{tikzcd} & N_1 \\ M \arrow[ur, "\varphi_1"] \arrow[dr, "\varphi_2"'] & \\ & N_2 \end{tikzcd}$$

then its colimit (pushout)

$$\begin{tikzcd} & N_1 \arrow[dr, "\sigma_1"] & \\ M \arrow[ur, "\varphi_1"] \arrow[dr, "\varphi_2"'] & & P \\ & N_2 \arrow[ur, "\sigma_2"'] & \end{tikzcd}$$

in PATH is inherited by PATH and the σ_i, $i = 1, 2$, are exact.

Proof. Clearly σ_1 and σ_2 are exact. To show that σ_1 is a path-morphism, suppose that $e \in E_P$, $v_1 \in V_{N_1}$ and $\sigma_1(v_1) = \partial_-(e)$. Since $e \in E_P$ there must be some $e_2 \in E_{N_1}$ or E_{N_2} (say E_{N_2}) such that $\sigma_2(e_2) = e$. Since $\sigma_2(\partial_-(e_2)) = \partial_-(\sigma_2(e_2)) = \partial_-(e) = \sigma_1(v_1)$, there must exist $w_1, \dots w_n \in E_M$ such that $\varphi_2(w_1) = \partial_-(e_2)$, $\varphi_1(w_n) = v_1$ and $\varphi_2(w_i) = \varphi_2(w_{i+1})$, $i = 2, 4, 6, \dots$, and $\varphi_1(w_i) = \varphi_1(w_{i+1})$, $i = 1, 3, 5, \dots$. Now we use the fact that φ_1 and φ_2 are path-morphisms to successively find edges $f_1, \dots f_n \in E_M$ such that $\partial_-(f_i) = w_i$ $1 \le i \le n$. Since $e_1 = \varphi_1(f_n) \in E_{N_1}$, $\partial_-(e_1) = \varphi_1(\partial_-(f_n))$ $= \varphi_1(w_n) = v_1$ and $\sigma_1(e_1) = \sigma_1(\varphi_1(f_n) = \sigma_2(\varphi_2(f_n)) = \sigma_2(\varphi_2(f_{n-1}))$ $= \cdots = \sigma_2(\varphi_2(f_1) = \sigma_2(e_2) = e$ and we are done.

EXAMPLE. The path-morphism of Fig. 4 is the coequalizer of the identity and the obvious nontrivial symmetry of the domain.

2. APPLICATIONS

2.1. *The Kernel and the Core*

Given a path-morphism $\varphi : M \to N$, its *kernel*, denoted $\ker(\varphi)$, is the maximal subnetwork of M with vertex set $V_{\ker(\varphi)} = \{v \in V_M : \omega(\varphi(v)) = \omega(v)\}$. (Thus $E_{\ker(\varphi)} = \{e \in E_M : \partial_\pm(e) \in V_{\ker(\varphi)}\}$). For a network, M, the *core of M*, denoted $C(M)$, will be $\cap\{\ker \varphi : \mathrm{dom}(\varphi) = M\}$. Thus $\mathcal{P}(C(M))$ will consist of all s–t paths in M which are locally minimal, i.e., cannot be reduced by any path-morphism.

A *core network* is a network such that $C(M) = M$. The set of all core networks determines a full subcategory of PATH all of whose morphisms are exact. In fact if M is a core network, any $\varphi: M \to N$ must be exact and N is a core network also. Thus C: PATH $\to$ PATH is a functor which preserves λ.

THEOREM. *$C(PATH)$, the category of all core networks, has pullbacks, pushouts, coequalizers, and local terminal objects (i.e., each component has a terminal object).*

Proof. The existence of pullbacks, pushouts and coequalizers follow directly from theorems of Section 1.2. The argument for local terminal objects is essentially the same as in the category FLOW [10], so we shall only sketch it here. Define a quasi-order "$\lesssim$" on C(PATH) by $N \lesssim M$ if there exists an (exact) path-morphism $\varphi: M \to N$. The corresponding equivalence relation "$\sim$" is isomorphism. The resulting partial order ("$\lesssim$" modulo "$\sim$") has the descending chain condition (no nontrivially infinite descending chains) and so each member N of C(PATH) majorizes some member T_N of a minimal class; in fact there must be a unique path-morphism $\varphi: N \to T_N$, since the coequalizer of two such maps would contradict the minimality of T_N. Similarly, if N_1 and N_2 are in the same component of C(PATH) repeated application of pushout will give a common lower bound to T_{N_1} and T_{N_2}, so $T_{N_1} \sim T_{N_2}$ choose any one of these minimal elements for the local terminal object.

Given a network M its *Dijkstra subnetwork*, $D(M)$, has vertex set

$$V_{D(M)} = \{v \in V_M : v \text{ lies on some minimum } s\text{–}t \text{ path}\},$$

and edge set

$$E_{D(M)} = \{e \in E_M : e \text{ lies on some minimum } s\text{–}t \text{ path}\}.$$

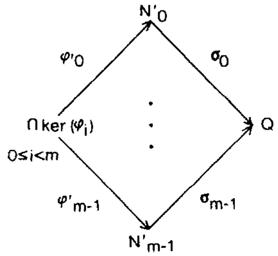
Note that $D(M)$ is a subnetwork of $C(M)$ and that D: PATH $\to$ PATH determines a λ-preserving functor into a subcategory all of whose morphisms are exact.

THEOREM. *$D(PATH)$, the category of all Dijkstra networks, has pullbacks, pushouts, coequalizers and local terminal objects.*

Proof. Same as for the previous theorem.

Suppose that $\varphi_i: M \to N_i$, $0 \le i < m$, each have a right inverse (in PATH) $\rho_i: N_i \to M$. The φ_i's, restricted to $\cap\{\ker(\varphi_i) : 0 \le i < m\}$ will be exact path-morphisms and so will have a coequalizer Q with $\sigma_i: N_i' \to Q$

which makes the diagram commute.

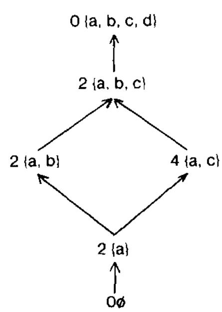


Then $\varphi = \lim_{k \to \infty} \sigma \circ [\prod_{i=0}^{k} (\rho_i \circ \varphi_i)] : M \to Q$ is defined, where $\prod_{i=0}^{k} \tau_i = \tau_k \circ \tau_{k-1} \circ \cdots \circ \tau_0$. Subscripts are computed modulo m and the limit exists in the trivial sense that for every $v \in V_M$ the sequence $\sigma \circ [\prod_{i=0}^{k} (\rho_i \circ \varphi_i)](v)$ is eventually constant. This follows from the observation that $[\prod_{i=0}^{k} (\rho_i \circ \varphi_i)](v)$ must eventually cycle and when it does then $[\prod_{i=0}^{k} (\rho_i \circ \varphi_i)](v)$ must be in $\bigcap_{0 \le i < m} \ker(\varphi_i)$. The value of $\rho \circ [\prod_{i=0}^{k} (\rho_i \circ \varphi_i)](v)$ will then not change with increasing k.

The construction of Q here as essentially the pushout of the φ_i's, but without the uniqueness of φ (its value may depend on the ordering of $\varphi_0, \varphi_1, \ldots, \varphi_{m-1}$) generalizes one of the principal results of [9], where for a set of stabilizing transformations, Q is represented as the lattice of lower sets in the stability order.

2.2. *The Edge-Sum Problem*

Returning now to the Edge-Sum Problem (ESP) of Section 0.2, we recall that symmetries of G induce symmetries of $N(G)$, the corresponding network. Since the symmetries of $N(G)$ have a coequalizer, the MPP on $N(G)$, which is equivalent to the ESP on G, is simplified by the systematic use of symmetry. When there are stabilizing reflections (see [9]), however, the simplification is more dramatic. For instance, when G is the graph of the square (Example 1), then $N(G)$ has $2^4 = 16$ vertices. The coequalizer of the symmetries of $N(G)$ induced by symmetries of G is



Obviously this network is reducible to a linear one (see Fig. 4) which is the "pushout" of all stabilizing transformations. For the graphs of higher-dimensional cubes, the "pushout" of all stabilizing transformations is not linear, but it is always stronger (a homomorph of) the coequalizer of all induced symmetries.

A second advantage for the "pushout" of stabilizing transformations is that they have a relatively efficient representation (as the Hasse diagram of the lattice of lower sets of the stability order).

As mentioned in Section 1.0, the networks of the graphs of all regular polytopes have been shown to have the K–H property except for two 4-dimensional polytopes, the 600-cell (120 vertices) and the 120-cell (600 vertices). It seemed reasonable to expect that these two would also have it and that there should be a uniform proof technique applying to all the regular solids. After searching in vain we put the process of computing the lattice of lower sets for the stability order of the 600-cell on the computer and found to our surprise that it is not so, it does *not* have the K–H property. Actually, a preliminary program written by a student, Patrick Jensen, had indicated this but there were some difficulties with the program and we were incredulous until we had written an independent program which verified it.

To see the amount of reduction which this "pushout" gives, note that $N(G)$ for the 600-cell would have $2^{120} \simeq 10^{36}$ vertices, whereas the corresponding lattice of lower sets has only about 900.

Using Dijkstra's algorithm on this network, we actually solved the ESP on the graph of the 600-cell and found $\min_{\varphi} \Sigma_{e \in E} \Delta_e(\varphi) = 12{,}620$. For details see [1]. The same computation for the 120-cell will give a reduced network of about 5 million vertices. Since completing this computation would be costly and we have no urgent questions which could be answered by it, we have not proceeded.

3. Conclusions and Comments

In the last section of the paper on the global theory of flows in networks [10], which the present paper has been modeled after, the question "What other problems are preserved by flow morphisms?" was broached and one very interesting answer was given (the Sperner–Erdös problem). Of course there are many variations on the max-flow problem which appear in the literature and the same question could have been addressed to them, but at the time there seemed to be no motivation for such a systematic approach and the questions which led to the present paper were more alluring. However, in reflecting on variations of the max-flow problem as well as other combinatorial problems which were candidates for investigation from the global point of view, it became apparent that there is some connection between polynomial algorithms and morphisms. That is, it seemed that problems which had "simple" solutions also had "nice" morphisms, and vice versa. The Sperner–Erdös problem was a good example of this phenomenon; having shown that it was preserved by the flow morphisms, Harper conjectured that it had a polynomial bounded solution and this was verified in [2]. The form of the solution, a reduction to the max-flow problem, also lends weight to the hypothesis that there is some connection between morphisms and algorithms.

In order to test this hypothesis independently of the circumstances which suggested it, we selected seven different variants of the minimum path problem (MPP) and asked for each one, (i) Does it have a tractable solution?, and (ii) Does it have a "nice" notion of morphism? The answers which we arrived at must be qualified to some extent; "tractability" means membership in P, the class of all problems having algorithms which produce a solution in time bounded by a polynomial in the size of the input. Being NP-complete has come to be regarded as strong evidence of intractability, but this conclusion depends upon the hypothesis that $P \neq NP$ (see [5] for details). Our minimal requirements for a "nice" notion of morphism were that it preserve the problem in both directions, that symmetries be morphisms and that the resulting category have coequalizers. Those problems for which we found such morphisms, the morphisms were mostly simple variations on path morphisms for which these properties could be easily verified. If such simple variations did not produce a satisfactory notion of morphism then we concluded that one did not exist. (This is clearly the weakest of the four possible answers.)

Even with these simple criteria, the routes by which final answers to our questions were arrived at were roundabout in several cases. One of the most instructive was the problem which we came to think of as the drill instructor's problem: A drill instructor has k recruits whom he wishes to run through an obstacle course (represented by a network N). The intermediate vertices represent the obstacles, and their weights the time it takes to traverse that obstacle. In order to avoid conflicts, he wishes to make their paths disjoint (except at s and t) and so that the sums of their elapsed times is a minimum. If $k = 1$ this reduces to the MPP. In general it is equivalent to computing

$$\min_{\substack{P_1, \ldots, P_k \in \mathcal{P}(N) \\ \partial(P_i) \cap \partial(P_j) = \{s, t\}}} \sum_{i=1}^{k} \omega(P_i).$$

The drill instructor's problem is not preserved by path morphisms since the images of disjoint paths may not be disjoint. However, this suggests characterizing the k disjoint paths of the recruits as integral flows of value k

with respect to the capacity $c(v) = 1$ for all $v \in V - \{s, t\}$. This makes the drill instructor's problem a special case of the min-cost flow problem found in [4]. One then observes that graph homomorphisms which are both path morphisms and flow morphisms (see [10]) preserve the min-cost flow problem and give a "nice" category. However, the "solution" which Ford Fulkerson give for the min-cost flow problem is not a polynomial bounded solution. It has the same defect which their "solution" for the max-flow problem has—it is only pseudo-polynomial. Dinic [3] has remedied this difficulty for the max-flow problem, but the same trick does not work for the min-cost flow problem. Seeing no way around this defect (note, however, that the Ford–Fulkerson algorithm does give a polynomial solution to the drill instructor's problem), we thought that the min-cost flow problem might be intractable and looked in the Garey–Johnson book. We found there the min-edgecost flow problem listed as NP-complete, but this problem is slightly different—the cost of a flow being $\Sigma_{f(e)>0}\omega(e)$, the sum of the unit costs for all edges in which $f(e) > 0$, rather than $\Sigma_{e\in E}\omega(e)f(e)$. In a footnote they refer to Lawler's book [11] for a solution to the min-cost flow problem. Lawler, following Fulkerson and Minty, presents a solution by the out-of-kilter method and a trick of successive approximations. The algorithm is evidently not very practical, but it is polynomial bounded.

As you can see then, the drill instructor's problem led us on a merry chase with several different answers to our questions being considered before arriving at a very satisfying pair of "yeses." Without further ado then we present the final answers to our questions about the min-path problem and its seven variations in Table I.

We see that the answers to our two questions are the same for every problem. That such a correspondence is due purely to coincidence seems unlikely, and the list could now be extended to include many other problems. The juxtaposition of several of the pairs of problems reveals further points to ponder: (i) generalizes (o) and the fact that it is still preserved by path morphisms is a simple observation, but the algorithm required to solve the extended problem is an order of magnitude more complicated than Dijkstra's. The pair (v) and (vi) is used by Garey and Johnson [5, p. 213] to illustrate how very similar problems can differ in their complexity. Problems (ii) and (vii) show that finding the kth smallest (or largest) member of a weighted set may be both easy and hard.

We are led then to believe that there is a relationship between "nice" morphisms and "simple" algorithms for combinatorial problems. Just what the relationship could be we have only a vague idea at the present time; there being a lot of structure in these examples which we have no coherent theory for. One thing which is clear from our experience is that the relationship, even in its present crude form, is useful in the analysis of algorithmic problems. We look forward to further research which will ellucidate this relationship and increase its usefulness.

TABLE I[a]

	Problem	"Nice" morphisms?	In P?
(o)	$\min_{P \in \mathcal{P}(N)} \omega(P), \omega(v) \in \mathbb{Z}^+$	Yes (PATH)	Yes (Dijkstra)
(i)	$\min_{P \in \mathcal{P}(N)} \omega(P), \omega(v) \in \mathbb{Z}$	Yes (PATH)	Yes (see [3, p. 15])
(ii)	$\min_{P \in \mathcal{P}(N)} \max_{v_1, \ldots, v_k \in \partial(P)} \min_{1 \le i \le k} \omega(v_i)$	Yes (PATH)	Yes
(iii)	$\min_{f \in \mathcal{F}_k(N)} \sum_{v \in V} \omega(v) \sum_{\partial_+(e) = v} f(e)$, where $\mathcal{F}_k(N)$ is the set of all flows on N with value k. This is the min-cost flow problem discussed earlier.	Yes	Yes
(iv)	$\#\mathcal{P}(N)$	No	No (see [5, 13])
(v)	$\max_{P \in \mathcal{P}(N)} \omega(P), \omega(P) \in \mathbb{Z}^+$	Yes (reverse "$\geq$" in PATH)	Yes
(vi)	$\max_{P \in \mathcal{P}_0(N)} \omega(P), \omega(v) \in \mathbb{Z}^+$ $\mathcal{P}_0(N) = \{\text{simple } s\text{–}t \text{ paths}\}$	No	(No (see [5, p. 213])
(vii)	$\min_{P_1, \ldots, P_k \in \mathcal{P}(N)} \max_{1 \le i \le k} \omega(P_i)$	No	No (see [5, p. 214])

[a]Recall that $\omega(P) = \Sigma_{v \in \partial(P)} \omega(v)$.

Note. Problem (v) is essentially contained in problem (i).

References

1. X. Berenguer and L. H. Harper, Estabilizacion y resolucion de algunos problemas combinatorios en grafos simetricos, *Questiío* **3**, no. 2 (1979), 105–117.
2. X. Berenguer, J. Diaz, and L. H. Harper, A solution of the Sperner-Erdös problem, preprint.
3. S. Evan, "Graph Algorithms," Computer Science Press, Potomac, Maryland, 1979.
4. L. R. Ford and D. R. Fulkerson, "Flows in Networks," Van Nostrand, Princeton, N.J., 1962.
5. M. R. Garey and D. S. Johnson, "Computers and Intractability, A Guide to the Theory of *NP*-Completeness," Freeman, San Francisco, 1979.
6. L. H. Harper, Optimal assignments of numbers to vertices, *SIAM J.* **12**, no. 1 (1964), 131–135.
7. L. H. Harper, Optimal numberings and isoperimetric problems on graphs, *J. Combinatorial Theory* **1**, no. 3 (1966), 385–393.
8. L. H. Harper, A necessary condition on minimal cube numberings, *J. Appl. Prob.* **4** (1967), 397–401.

9. L. H. HARPER, Stabilization and the edge-sum problem, *Ars Combinatoria* **4** (1977), 225–270.
10. L. H. HARPER, The global theory of flows in networks, *Advances Appl. Math.* **1** (1980), 158–181.
11. E. L. LAWLER, "Combinatorial Optimization: Networks and Matroids," Holt, Rhinehart & Winston, New York, 1976.
12. S. MACLANE, "Categories for the Working Mathematician," Springer-Verlag, New York, 1971.
13. L. VALIENT, "The complexity of Enumeration and Reliability Problems, Report No. CSR-15-77, Computer Science Dept., University of Edinburgh, Edinburgh, Scotland.